数理解析研究所講究録
第 1893 巻 2014 年 35-45

Extensions of relative operator entropies and operator α-divergence

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1. Generalized relative operator entropy

Throughout this note, A and B are positive operators on a Hilbert space. In [2](cf.[3,15], we gave the relative operator entropy by

$$S(A|B)=\lim_{r\rightarrow 0}\frac{A\ \sharp_{r}\ B-A}{r}=A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}},$$

where the notation $A\ \sharp_{r}\ B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r}A^{\frac{1}{2}},\quad r\in(0,\ 1)$, is the geometric operator mean in the sence of Kubo-Ando [13]. Tsallis relative operator entropy is defined by Yanagi, Kuriyama and Furuichi in [16] as follows.

$$T_{r}(A|B)=\frac{A\ \sharp_{r}\ B-A}{r}.$$

As a generalization of $S(A|B)$, Furuta [7] has given the following:

$$S_{r}(A|B)=\lim_{\epsilon\rightarrow 0}\frac{A\ \natural_{r+\epsilon}\ B-A\ \natural_{r}\ B}{\epsilon}=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}=(A\natural_{r}B)\cdot A^{-1}\cdot S(A|B),$$

here we call $A\ \natural_{r}\ B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r}A^{\frac{1}{2}}$, for $r\in \mathbf{R}$, a path going through A and B, which coincides with $A\ \sharp_{r}\ B$ if $r\in[0,\ 1]$.

Theorem 1.([8]). For $A>0,\ B>0,\ S_{r}(A|B)$ is monotone increasing for $r\in \mathbf{R}$, and the following holds.

$$S_{r}(A|B)\leq\frac{A\ \natural_{q}\ B-A\ \natural_{r}\ B}{q-r}\leq S_{q}(A|B)\quad \text{for } q,\ r\in \mathbf{R},\ q>r. \tag{1}$$

Especially, in the case $r=0$ and $0<q<1$, (1) is expressed as follows:

$$S(A|B)\leq\frac{A\ \sharp_{q}\ B-A}{q}=T_{q}(A|B)\leq S_{q}(A|B). \tag{2}$$

To prove Theorem 1, we use the next Lemma.

Lemma 2.([8]). Let $a>0$. Then the following holds for $q,\ r\in \mathbf{R}$.

$$a^{r}\log a\leq\frac{a^{q}-a^{r}}{q-r}\leq a^{q}\log a,\quad \text{for}\quad q>r.$$

Since a^{t} is convex function, this is easily given, but we give an elementary proof.

Proof. We show this inequality as follows:

$$\frac{a^{q}}{a^{r}}\log\frac{a^{q}}{a^{r}}=-\frac{a^{q}}{a^{r}}\log\frac{a^{r}}{a^{q}}\geq-\frac{a^{q}}{a^{r}}(\frac{a^{r}}{a^{q}}-1)=\frac{a^{q}}{a^{r}}-1\geq\log\frac{a^{q}}{a^{r}},$$

that is,

$$a^q(\log a^q - \log a^r) \geq a^q - a^r \geq a^r(\log a^q - \log a^r).$$

So we have

$$(q-r)a^q \log a \geq a^q - a^r \geq (q-r)a^r \log a.$$

Generalizing Theorem 1, we give the following basic relations in this note.

Theorem 3.([8,11]). For $A>0,\ B>0$, the following hold.

(1) If $0<r<1$, then

$$S(A|B) \leq T_r(A|B) \leq S_r(A|B) \leq -T_{1-r}(B|A) \leq -S(B|A) = S_1(A|B).$$

(2) If $n<r<n+1$, then

$$S_n(A|B) \leq \frac{A \natural_r B - A \natural_n B}{r-n} \leq S_r(A|B) \leq \frac{A \natural_{n+1} B - A \natural_r B}{n+1-r} \leq S_{n+1}(A|B)$$

$\Longleftrightarrow$

$$\begin{aligned} & (BA^{-1})^n S(A|B) \leq (BA^{-1})^n T_{r-n}(A|B) \leq (BA^{-1})^n S_{r-n}(A|B) \\ \leq \ & -(BA^{-1})^n T_{n+1-r}(B|A) \leq -(BA^{-1})^n S(B|A) = (BA^{-1})^n S_1(A|B). \end{aligned}$$

$\Longleftrightarrow$

$$\begin{aligned} & S(A \natural_n B|A \natural_{n+1} B) \leq T_{r-n}(A \natural_n B|A \natural_{n+1} B) \leq S_{r-n}(A \natural_n B|A \natural_{n+1} B) \\ \leq \ & -T_{n+1-r}(A \natural_{n+1} B|A \natural_n B) \leq S_1(A \natural_n B|A \natural_{n+1} B). \end{aligned}$$

The following properties of $S_r(A|B)$ and $T_r(A|B)$ are important in our discussion.

Lemma 4.([8,11]). Let n be an integer. Then, for $r \in \mathbf{R}$, $S_r(A|B)$ has the following properties:

(1) $$S_r(A|B) = -S_{1-r}(B|A) = BS_{r-1}(B^{-1}|A^{-1})B = -AS_{-r}(A^{-1}|B^{-1})A,$$

(2) $$S_n(A|B) = (BA^{-1})^n S(A|B) = S(A|B)(A^{-1}B)^n,$$

(3) $$S_r(A|B) = (A \natural_r B)\cdot A^{-1} \cdot S(A|B) = S(A \natural_r B|A \natural_{r+1} B),$$

(4) $$\tfrac{A \natural_r B - A \natural_n B}{r-n} = (BA^{-1})^n T_{r-n}(A|B) = T_{r-n}(A|B)(A^{-1}B)^n,$$

(5) $$\tfrac{A \natural_{n+1} B - A \natural_r B}{n+1-r} = -(BA^{-1})^n T_{n+1-r}(B|A) = -T_{n+1-r}(B|A)(A^{-1}B)^n.$$

Proof. (1) is given as follows:

$$\begin{aligned} S_r(A|B) &= A^{\frac12}(A^{-\frac12}BA^{-\frac12})^r(\log A^{-\frac12}BA^{-\frac12})A^{\frac12} \\ &= -B^{\frac12}B^{-\frac12}A^{\frac12}(A^{\frac12}B^{-1}A^{\frac12})^{-r}(\log A^{\frac12}B^{-1}A^{\frac12})A^{\frac12}B^{-\frac12}B^{\frac12} \\ &= -B^{\frac12}(B^{-\frac12}AB^{-\frac12})^{-r}(\log B^{-\frac12}AB^{-\frac12})(B^{-\frac12}AB^{-\frac12})B^{\frac12} \\ &= -B^{\frac12}(B^{-\frac12}AB^{-\frac12})^{-r+1}(\log B^{-\frac12}AB^{-\frac12})B^{\frac12} = -S_{-r+1}(B|A), \\ & \textit{or} \\ &= B^{\frac12}(B^{\frac12}A^{-1}B^{\frac12})^{r-1}(\log B^{\frac12}A^{-1}B^{\frac12})B^{\frac12} \\ &= BB^{-\frac12}(B^{\frac12}A^{-1}B^{\frac12})^{r-1}(\log B^{\frac12}A^{-1}B^{\frac12})B^{-\frac12}B = BS_{r-1}(B^{-1}|A^{-1})B. \end{aligned}$$

The last equation is shown by the similar way.
(2) is shown as follows:

$$\begin{aligned} S_n(A|B) &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \\ &= (BA^{-1})^n A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = (BA^{-1})^n S(A|B). \end{aligned}$$

(3) follows from the definitions of $S_r(A|B)$ and $S(A \natural_r B|A \natural_{r+1} B)$.

$$S_r(A|B) = \lim_{\epsilon \to 0} \frac{A \natural_{r+\epsilon} B - A \natural_r B}{\epsilon} = (A \natural_r B) \cdot A^{-1} \cdot S(A|B)$$

$$S(A \natural_r B|A \natural_{r+1} B) = \lim_{\epsilon \to 0} \frac{(A \natural_r B) \cdot A \cdot (A \sharp_\epsilon B - A)}{\epsilon} = (A \natural_r B) \cdot A^{-1} \cdot S(A|B).$$

(4) is shown as follows:

$$\begin{aligned} \frac{A \natural_r B - A \natural_n B}{r-n} &= \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n\{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r-n} - I\}A^{\frac{1}{2}}}{r-n} \\ &= \frac{(BA^{-1})^n A^{\frac{1}{2}}\{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r-n} - I\}A^{\frac{1}{2}}}{r-n} \\ &= \frac{(BA^{-1})^n(A \sharp_{r-n} B - A)}{r-n} = (BA^{-1})^n T_{r-n}(A|B), \end{aligned}$$

and (5) is also given similarly,

$$\begin{aligned} \frac{A \natural_{n+1} B - A \natural_r B}{n+1-r} &= \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{n+1}A^{\frac{1}{2}} - A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}}}{n+1-r} \\ &= \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r-n}\}A^{\frac{1}{2}}}{n+1-r} \\ &= \frac{(BA^{-1})^n A^{\frac{1}{2}}\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r-n}\}A^{\frac{1}{2}}}{n+1-r} \\ &= \frac{(BA^{-1})^n(B - A \sharp_{r-n} B)}{n+1-r} = \frac{(BA^{-1})^n(B - B \sharp_{n+1-r} A)}{n+1-r} \\ &= -(BA^{-1})^n T_{n+1-r}(B|A) \end{aligned}$$

2. Extension of Tsallis relative operator entropy

The following is called the power mean [4,12],

$$\begin{aligned} A \sharp_{t,r} B &= A^{\frac{1}{2}}\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}}A^{\frac{1}{2}}, \quad t \in [0,\ 1], \quad r \in [-1,\ 1], \\ &= A \natural_{\frac{1}{r}} \{A \nabla_t (A \natural_r B)\}. \end{aligned}$$

It is known that $A \sharp_{t,1} B = A \nabla_t B$, $A \sharp_{t,0} B = A \sharp_t B$ and $A \sharp_{t,-1} B = A \Delta_t B$, where $A \nabla_t B = (1-t)A + tB$ is the arithmetic operator mean and $A \Delta_t B = (A^{-1} \nabla_t B^{-1})^{-1}$ is the harmonic operator mean. Using this, we can define an extension of Tsallis relative operator entropy (cf.[10]),

$$T_{t,r}(A|B) = \frac{A \sharp_{t,r} B - A}{t}.$$

Relations between $T_{t,r}(A|B)$, $T_t(A|B)$, $T_r(A|B)$ and $S(A|B)$ are given by the following diagram.

$$\begin{array}{ccc} T_{t,r}(A|B) & \longrightarrow_{t \to 0} & T_r(A|B) \\ \downarrow_{r \to 0} & & \downarrow_{r \to 0} \\ T_t(A|B) & \longrightarrow_{t \to 0} & S(A|B) \end{array}$$

Since the corresponding function to the power mean is $p(t,r)=\{1-t+ta^r\}^{\frac{1}{r}}$ and $\frac{\partial}{\partial t}p(t,r)=\{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r}$, we can define

$$\begin{aligned} S_{t,r}(A|B) &= A^{\frac{1}{2}}\Big(\{I+t((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r-I)\}^{\frac{1}{r}-1}\cdot\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r-I}{r}\Big)A^{\frac{1}{2}}, \quad r\neq 0, \\ &= A\natural_{\frac{1-r}{r}}\{A\nabla_t(A\natural_r B)\}\cdot A^{-1}\cdot\frac{A\natural_r B-A}{r}. \\ \lim_{r\to 0}S_{t,r}(A|B) &= S_{t,0}(A|B)=S_t(A|B). \quad \text{This will be shown later.} \end{aligned}$$

We call this an expanded relative operator entropy. Then we can show an expanded form of Theorem 3 (2) as follows [9]:

Theorem 6. For $A>0$ and $B>0$,

$$S_{0,r}(A|B)\le T_{t,r}(A|B)\le S_{t,r}(A|B)\le -T_{1-t,r}(B|A)\le S_{1,r}(A|B)$$

holds for $t\in[0,\ 1]$ and $r\in[-1,\ 1]$.

Proof. (1) First we show $S_{0,r}(A|B)\le T_{t,r}(A|B)$. Since $S_{0,r}(A|B)=T_r(A|B)$, we have only to show that

$$\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-1}{t}\ge\frac{a^r-1}{r}.$$

Let

$$f(t)=\{1+t(a^r-1)\}^{\frac{1}{r}}-1-t\frac{a^r-1}{r}.$$

Then we have

$$\frac{d}{dt}f(t)=\{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r}-\frac{a^r-1}{r}=\frac{\partial}{\partial t}p(t,r)-\frac{a^r-1}{r}$$

and

$$\frac{\partial^2}{\partial t^2}p(t,r)=\{1+t(a^r-1)\}^{\frac{1}{r}-2}\cdot\frac{(a^r-1)^2(1-r)}{r^2}\ge 0.$$

So $\frac{\partial}{\partial t}p(t,r)$ is an increasing function for $t\in[0,\ 1]$ and $\frac{\partial}{\partial t}p(t,r)|_{t=0}=\frac{a^r-1}{r}$. Since $\frac{df(t)}{dt}\ge 0$ by $\frac{df(0)}{dt}=0$, we have $f(t)$ is increasing for $t\in[0,\ 1]$ and $f(0)=0$, so we have $\{1+t(a^r-1)\}^{\frac{1}{r}}-1\ge t\frac{a^r-1}{r}$, that is, $\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-1}{t}\ge\frac{a^r-1}{r}$ and $S_{0,r}(A|B)\le T_{t,r}(A|B)$.

(2) Second, we show $T_{t,r}(A|B)\le S_{t,r}(A|B)\le -T_{t,r}(B|A)$. It is sufficient to show

$$\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-1}{t}\le\{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r}.$$

Since

$$\frac{d^2}{dt^2}f(t)=\frac{\partial^2}{\partial t^2}p(t,\ r)\ge 0, \quad \text{for } t\in[0,\ 1] \text{ and } t\in[-1,\ 1],$$

$f(t)$ and $p(t,r)$ are convex for t on $[0,\ 1]$.

Since a function $f(t)$ is convex, then, for $a,\ h>0$,

$$f(a)=f(\frac{a}{a+h}(a+h)+\frac{h}{a+h}0)\le\frac{a}{a+h}f(a+h)+\frac{h}{a+h}f(0),$$

so that

$$\frac{f(a+h)-f(a)}{h} \geq \frac{f(a)-f(0)}{a}.$$

Hence we have

$$\frac{p(t+h,\ r)-p(t,\ r)}{h} \geq \frac{p(t,\ r)-p(0,\ r)}{t}, \text{ for } \forall h>0,$$

that is,

$$\frac{(1+(t+h)(a^r-1))^{\frac{1}{r}}-(1+t(a^r-1))^{\frac{1}{r}}}{h} \geq \frac{(1+t(a^r-1))^{\frac{1}{r}}-1}{t}, \text{ for } \forall h>0,$$

and

$$\begin{aligned} &\lim_{h\to+0} \frac{(1+(t+h)(a^r-1))^{\frac{1}{r}}-(1+t(a^r-1))^{\frac{1}{r}}}{h} \\ = &\ \{1+t(a^r-1)\}^{\frac{1}{r}-1} \cdot \frac{a^r-1}{r} \geq \frac{(1+t(a^r-1))^{\frac{1}{r}}-1}{t}. \end{aligned}$$

So we can conclude

$$S_{t,r}(A|B) \geq T_{t,r}(A|B).$$

(3) Next we show $S_{t,r}(A|B) \leq -T_{1-t,r}(B|A)$. Since

$$\begin{aligned} T_{1-t,r}(B|A) &= \frac{B\ \natural_{1-t,r}\ A-B}{1-t} = \frac{A\ \natural_{t,r}\ B-B}{1-t} \\ &= \frac{A^{\frac{1}{2}}\left(\{I+t((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r-I)\}^{\frac{1}{r}}-(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)A^{\frac{1}{2}}}{1-t}, \end{aligned}$$

the function corresponding to $T_{1-t,r}(B|A)$ is $\dfrac{p(t,\ r)-a}{1-t}$.
For a convex function $f(t)$,

$$f(t)=f\left(\frac{1-t}{1-t+h}(t-h)+\frac{h}{1-t+h}\cdot 1\right) \leq \frac{1-t}{1-t+h}f(t-h)+\frac{h}{1-t+h}f(1).$$

So we have

$$\frac{f(t)-f(t-h)}{h} \leq \frac{f(1)-f(t)}{1-t}.$$

For $p(t,\ r)=(1+t(a^r-1))^{\frac{1}{r}}$,

$$\frac{p(t,\ r)-p(t-h,\ r)}{h} \leq \frac{p(1,\ r)-p(t,\ r)}{1-t},$$

that is,

$$\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-\{1+(t-h)(a^r-1)\}^{\frac{1}{r}}}{h} \leq \frac{a-\{1+t(a^r-1)\}^{\frac{1}{r}}}{1-t}.$$

Since

$$\lim_{h\to 0} \frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-\{1+(t-h)(a^r-1)\}^{\frac{1}{r}}}{h} = \frac{\partial}{\partial t}p(t,\ r) = \{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r},$$

we have

$$\{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r} \leq \frac{a-\{1+t(a^r-1)\}^{\frac{1}{r}}}{1-t}.$$

So we conclude $S_{t,r}(A|B) \le -T_{1-t,r}(B|A)$.

(4) Finally, we see $-T_{1-t,r}(B|A) \le S_{1,r}(A|B)$, it is sufficient to show that

$$-\frac{\{1+t(a^r-1)\}^{\frac{1}{r}}-a}{1-t} \le \frac{a-a^{1-r}}{r}.$$

This is equivalent to

$$\{1+t(a^r-1)\}^{\frac{1}{r}}-a \ge -(1-t)\frac{a-a^{1-r}}{r}.$$

Let

$$g(t) = \{1+t(a^r-1)\}^{\frac{1}{r}}-a+(1-t)\frac{a-a^{1-r}}{r},$$

then for $t \in [0,\ 1]$,

$$\frac{d}{dt}g(t) = \{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r}-\frac{a-a^{1-r}}{r}.$$

As we showed above $\frac{\partial}{\partial t}p(t,r) = \{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r}$ is an increasing function for $t \in [0,\ 1]$ and $\frac{d}{dt}g(t)|_{t=1} = 0$. Hence $\frac{d}{dt}g(t) \le 0$ for $t \in [0,\ 1]$. So the function $g(t)$ is a decreasing function and $g(1) = 0$, that is, $g(t) \ge 0$ for $t \in [0,\ 1]$. Hence we have the conclusion $-T_{1-t,r}(B|A) \le S_{1,r}(A|B)$.

Remark. **([9]).** The following relations hold, so Theorem 3 (1) is the case $r=0$ in Theorem 6. For $A>0$, $B>0$ and $t \in [0,\ 1]$, $r \in [-1,\ 1]$, the followings hold:

(1) $$\lim_{r\to 0} S_{t,r}(A|B) = S_t(A|B),$$

(2) $$S_{0,r}(A|B) = T_r(A|B),$$

(3) $$S_{1,r}(A|B) = -T_r(B|A),$$

(4) $$\lim_{r\to 0} S_{0,r}(A|B) = S(A|B),$$

(5) $$\lim_{r\to 0} S_{1,r}(A|B) = S_1(A|B).$$

Proof. Since

$$\lim_{r\to 0}\{1+t(a^r-1)\}^{\frac{1}{r}-1}\cdot\frac{a^r-1}{r} = a^t\log a,$$

we have

$$\begin{aligned}&\lim_{r\to 0} A^{\frac{1}{2}}\left(\{I+t((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r-I)\}^{\frac{1}{r}-1}\cdot\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r-I}{r}\right)A^{\frac{1}{2}}\\ &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = S_t(A|B).\end{aligned}$$

Moreover,

$$\frac{\partial}{\partial t}p(t,\ r)|_{t=0} = \frac{a^r-1}{r},\quad \frac{\partial}{\partial t}p(t,\ r)|_{t=1} = \frac{a-a^{1-r}}{r}$$

and their limits are known that

$$\lim_{r\to 0}\frac{a^r-1}{r} = \log a,\quad \lim_{r\to 0}\frac{a-a^{1-r}}{r} = a\log a,$$

so we have

$$\lim_{r\to 0} A^{\frac{1}{2}} \left(\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I}{r} \right) A^{\frac{1}{2}} = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = S(A|B),$$

and

$$\begin{aligned}&\lim_{r\to 0} A^{\frac{1}{2}} \left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-r}}{r} \right) A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = S_1(A|B).\end{aligned}$$

We can give an expanded version of Theorem 3 (2) also.

Theorem 7. For $A > 0$, $B > 0$, $t \in [0,\ 1]$, $r \in [-1,\ 1]$ and an integer n, the following equivalence holds:

$$\begin{aligned}(1)\qquad & S_{0,r}(A \natural_n B|A \natural_{n+1} B) \le T_{t,r}(A \natural_n B|A \natural_{n+1} B) \le S_{t,r}(A \natural_n B|A \natural_{n+1} B) \\ \le\ & -T_{1-t,r}(A \natural_{n+1} B|A \natural_n B) \le S_{1,r}(A \natural_n B|A \natural_{n+1} B)\end{aligned}$$

$\Longleftrightarrow$

$$\begin{aligned}(2)\qquad & (BA^{-1})^n \cdot S_{0,r}(A|B) \le (BA^{-1})^n \cdot T_{t,r}(A|B) \le (BA^{-1})^n \cdot S_{t,r}(A|B) \\ \le\ & -(BA^{-1})^n \cdot T_{1-t,r}(B|A) \le (BA^{-1})^n \cdot S_{1,r}(A|B).\end{aligned}$$

Proof. (1) follows from Theorem 6 and we can obtain (2) by using the next lemma.

Lemma 8. For $A > 0$, $B > 0$, $t \in [0,\ 1]$, $r \in [-1,\ 1]$ and an integer n, the following hold.

$$(1)\qquad S_{0,r}(A \natural_n B|A \natural_{n+1} B) = (BA^{-1})^n \cdot S_{0,r}(A|B) = S_{0,r}(A|B) \cdot (A^{-1}B)^n,$$

$$(2)\qquad T_{t,r}(A \natural_n B|A \natural_{n+1} B) = (BA^{-1})^n \cdot T_{t,r}(A|B) = T_{t,r}(A|B) \cdot (A^{-1}B)^n,$$

$$(3)\qquad S_{t,r}(A \natural_n B|A \natural_{n+1} B) = (BA^{-1})^n \cdot S_{t,r}(A|B) = S_{t,r}(A|B) \cdot (A^{-1}B)^n,$$

$$(4)\qquad T_{1-t,r}(A \natural_{n+1} B|A \natural_n B) = (BA^{-1})^n \cdot T_{1-t,r}(B|A) = T_{1-t,r}(B|A) \cdot (A^{-1}B)^n,$$

$$(5)\qquad S_{1,r}(A \natural_n B|A \natural_{n+1} B) = (BA^{-1})^n \cdot S_{1,r}(A|B) = S_{1,r}(A|B) \cdot (A^{-1}B)^n.$$

Proof. Since

$$(A \natural_n B) \sharp_{t,r} (A \natural_{n+1} B) = (BA^{-1})^n \cdot (A \sharp_{t,r} B) = (A \sharp_{t,r} B) \cdot (A^{-1}B)^n,$$

and

$$\begin{aligned}A \sharp_{t,r} B &= A \natural_{\frac{1}{r}} (A \nabla_t (A \natural_r B)) \\ &= A \natural_{\frac{1}{r}} (A \nabla_t (B \natural_{1-r} A)) \\ &= B^{\frac{1}{2}}((B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) \natural_{\frac{1}{r}} \{(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) \nabla_t (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})\}^{1-r})B^{\frac{1}{2}} \\ &= B^{\frac{1}{2}}\{(1-t)(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^r + tI\}^{\frac{1}{r}}B^{\frac{1}{2}} = B \sharp_{1-t,r} A,\end{aligned}$$

we have the following:

$$\begin{aligned}
(1)\ S_{0,r}(A \natural_n B|A \natural_{n+1} B) &= T_r(A \natural_n B|A \natural_{n+1} B) = \frac{(A \natural_n B) \natural_r (A \natural_{n+1} B) - (A \natural_n B)}{r} \\
&= \frac{(A \natural_n B) \cdot A^{-1} \cdot (A \natural_r B - A)}{r} = (A \natural_n B) \cdot A^{-1} \cdot \frac{A \natural_r B - A}{r} \\
&= (BA^{-1})^n \cdot T_r(A|B) = (BA^{-1})^n \cdot S_{0,r}(A|B). \\
(2)\ T_{t,r}(A \natural_n B|A \natural_{n+1} B) &= \frac{(A \natural_n B) \sharp_{t,r} (A \natural_{n+1} B) - A \natural_n B}{t} \\
&= \frac{(A \natural_n B) \cdot A^{-1} \cdot (A \sharp_{t,r} B) - A \sharp_n B}{t} \\
&= (A \natural_n B) \cdot A^{-1} \cdot \frac{A \sharp_{t,r} B - A}{t} = (BA^{-1})^n \cdot T_{t,r}(A|B). \\
(3)\ S_{t,r}(A \natural_n B|A \natural_{n+1} B) &= (A \natural_n B) \natural_{\frac{1-r}{r}} \{(A \natural_n B) \nabla_t ((A \natural_n B) \natural_r (A \natural_{n+1} B))\} \cdot (A \natural_n B)^{-1} \\
&\quad \times \frac{(A \natural_n B) \natural_r (A \natural_{n+1} B) - A \natural_n B}{r} \\
&= (A \natural_n B) \natural_{\frac{1-r}{r}} \{(A \natural_n B) \nabla_t ((A \natural_n B) \natural_r (A \natural_{n+1} B))\} \cdot (A \natural_n B)^{-1} \\
&\quad \times (A \natural_n B) \cdot A^{-1} \cdot \frac{A \natural_r B - A}{r} \\
&= \Big((A \natural_n B) \natural_{\frac{1-r}{r}} ((A \natural_n B) \cdot A^{-1} \cdot (A \nabla_t (A \natural_r B)))\Big) \cdot A^{-1} \cdot \frac{A \natural_r B - A}{r} \\
&= (A \natural_n B) \cdot A^{-1} \{A \natural_{\frac{1-r}{r}} (A \nabla_t (A \natural_r B))\} \cdot A^{-1} \cdot \frac{A \natural_r B - A}{r} \\
&= (BA^{-1})^n \cdot S_{t,r}(A|B). \\
(4)\ T_{1-t,r}(A \natural_{n+1} B|A \natural_n B) &= \frac{(A \natural_{n+1} B) \sharp_{1-t,r} (A \natural_n B) - A \natural_{n+1} B}{1-t} \\
&= \frac{(A \natural_n B) \sharp_{t,r} (A \natural_{n+1} B) - A \natural_{n+1} B}{1-t} \\
&= \frac{(A \natural_n B) \cdot A^{-1} \cdot (A \sharp_{t,r} B - B)}{1-t} \\
&= (A \natural_n B) \cdot A^{-1} \cdot \frac{B \sharp_{1-t,r} A - B}{1-t} = (BA^{-1})^n \cdot T_{1-t,r}(B|A). \\
(5)\ S_{1,r}(A \natural_n B|A \natural_{n+1} B) &= -T_r(A \natural_{n+1} B|A \natural_n B) = -\frac{(A \natural_{n+1} B) \natural_r (A \natural_n B) - A \natural_{n+1} B}{r} \\
&= -\frac{(BA^{-1})^n (A \natural_{1-r} B) - B}{r} = -(BA^{-1})^n T_r(B|A) = -(BA^{-1})^n S_{1,r}(A|B).
\end{aligned}$$

3. Operator divergence

Petz introduced the Bregman operator divergence [14]: For an operator convex function F,

$$D_{[F]}(A|B) = F(A) - F(B) - \lim_{t \to 0} \frac{F(B + t(A-B)) - F(B)}{t}.$$

Another operator version of the Bregman divergence was also given by him,

$$D_0(A|B) = B - A - S(A|B).$$

Our interpretation of $D_0(A|B)$ has been $D_0(A|B) = \lim_{t \to +0} \frac{A \nabla_t B - A \sharp_t B}{t}$and $D_1(A|B)$ is also given by $D_1(A|B) = \lim_{t \to 1-0} \frac{A \nabla_t B - A \sharp_t B}{t-1}$.

But we are known the existence of α-divergence. The α-divergence is defined by Amari [1]. For probability distributions $p = (p_1, \cdots, p_n)$ and $q = (q_1, \cdots, q_n)$,

$$D_\alpha(p|q) = \frac{4}{1-\alpha^2}\Big(1 - \sum_{i=1}^n p_i^{\frac{1-\alpha}{2}} q_i^{\frac{1+\alpha}{2}}\Big), \quad \alpha \neq \pm 1,$$

where p_i, $q_i \geq 0$ for $i = 1, \cdots, n$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$. If we put $t = \frac{1+\alpha}{2}$, then the α-divergence can be expressed as follows:

$$D_t(p|q) = \frac{1}{t(1-t)} \sum_{i=1}^{n} \{(1-t)p_i + tq_i - p_i^{1-t} q_i^{t}\}, \quad t \neq 0, 1.$$

The operator version of the α-divergence is given in [5,6] as follows:

$$D_\alpha(A|B) = \frac{A \nabla_\alpha B - A \natural_\alpha B}{\alpha(1-\alpha)}, \quad \text{for } 0 < \alpha < 1.$$

This also satisfies the following:

$$D_0(A|B) = \lim_{\alpha \to 0} \frac{A \nabla_\alpha B - A \natural_\alpha B}{\alpha(1-\alpha)} = B - A - S(A|B),$$

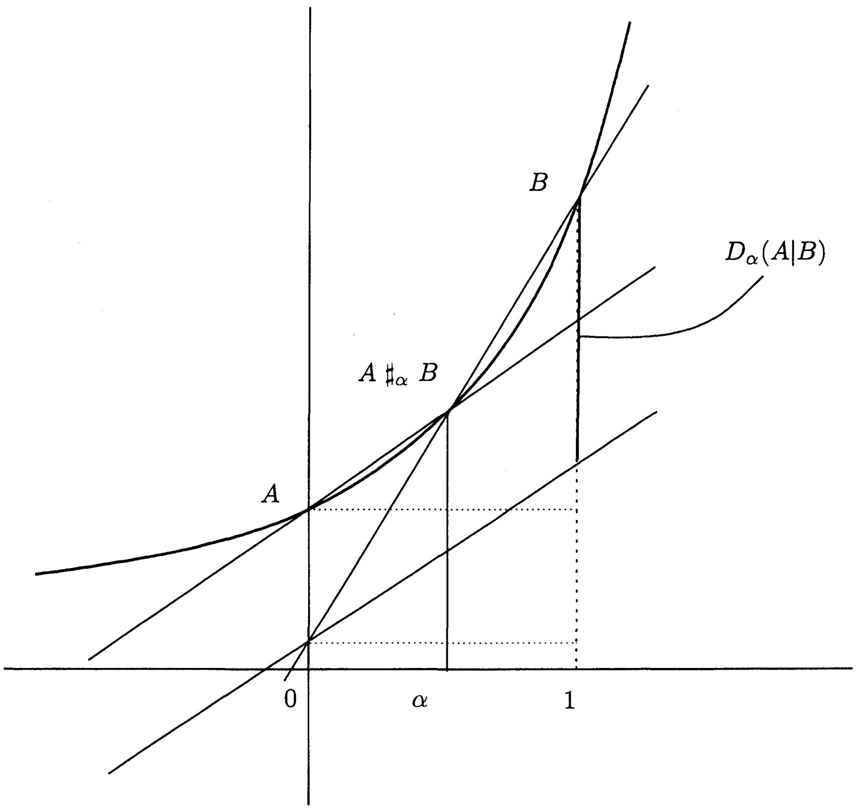
$$D_1(A|B) = \lim_{\alpha \to 1} \frac{A \nabla_\alpha B - A \natural_\alpha B}{\alpha(1-\alpha)} = A - B - S(B|A).$$

We can combine the Tsallis relative operator entropy and this operator α-divergence as follows:

Theorem 9.

$$D_\alpha(A|B) = -\big(T_\alpha(A|B) + T_{1-\alpha}(B|A)\big), \quad \text{for } 0 < \alpha < 1.$$

This can be shown as the following figure:



Proof of Theorem 9.

$$\begin{aligned} D_\alpha(A|B) &= \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)} = \frac{(1-\alpha)A + tB - (1-\alpha)(A \sharp_\alpha B) - \alpha(A \sharp_\alpha B)}{\alpha(1-\alpha)} \\ &= -\frac{A \sharp_\alpha B - A}{\alpha} - \frac{A \sharp_\alpha B - B}{1-\alpha} = -\frac{A \sharp_\alpha B - A}{\alpha} - \frac{B \sharp_{1-\alpha} A - B}{1-\alpha} \\ &= = -\big(T_\alpha(A|B) + T_{1-\alpha}(B|A)\big) \end{aligned}$$

So we can propose an extension of the operator α-divergence as follows:

$$D_{t,r}(A|B) = -\big(T_{t,r}(A|B) + T_{1-t,r}(B|A)\big), \quad \text{for } t \in [0,\ 1],\ r \in [-1,\ 1].$$

Theorem 10.

$$D_{t,r}(A|B) = \frac{A \nabla_t B - A \sharp_{t,r} B}{t(1-t)}, \quad t \in (0,\ 1).$$

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